

C -eigenvalues intervals for Piezoelectric-type tensors

Chaoqian Li^a, Yaotang Li^{a,*}

^a*School of Mathematics and Statistics, Yunnan University, Kunming, P. R. China
650091*

Abstract

C -eigenvalues of piezoelectric-type tensors which are real and always exist, are introduced by Chen et al. [1]. And the largest C -eigenvalue for the piezoelectric tensor determines the highest piezoelectric coupling constant. In this paper, we give two intervals to locate all C -eigenvalues for a given Piezoelectric-type tensor. These intervals provide upper bounds for the largest C -eigenvalue. Numerical examples are also given to show the corresponding results.

Keywords: Piezoelectric tensors, C -eigenvalues, Interval.

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1. Introduction

Piezoelectric-type tensors are introduced by Chen et al. in [1] as a subclass of third order tensors which have extensive applications in physics and engineering [2, 3, 5, 6, 7, 9]. The class of Piezoelectric tensors, as the subclass of Piezoelectric-type tensors of dimension three, plays the key role in Piezoelectric effect and converse Piezoelectric effect [1].

Definition 1. [1, Definition 2.1] Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ be a third-order n dimensional real tensor. If the later two indices of \mathcal{A} are symmetric, i.e., $a_{ijk} = a_{ikj}$ for all $j \in N$ and $k \in N$ where $N := \{1, 2, \dots, n\}$, then \mathcal{A} is called a piezoelectric-type tensor.

*Corresponding author.

Email addresses: lichaoqian@ynu.edu.cn (Chaoqian Li), liyaotang@ynu.edu.cn (Yaotang Li)

To explore more properties related to piezoelectric effect and converse piezoelectric effect in solid crystal, Chen et al. in [1] introduced C -eigenvalues and C -eigenvectors for Piezoelectric-type tensors, and shown that the largest C -eigenvalue corresponds to the electric displacement vector with the largest 2-norm in the piezoelectric electronic effect under unit uniaxial stress [1, 2, 8].

Definition 2. [1, Definition 2.2] Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ be a piezoelectric-type tensor. If there exist a scalar $\lambda \in \mathbb{R}$, vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ satisfying the following system

$$\mathcal{A}yy = \lambda x, \quad x\mathcal{A}y = \lambda y, \quad x^T x = 1 \text{ and } y^T y = 1, \quad (1)$$

where $\mathcal{A}yy \in \mathbb{R}^n$ and $x\mathcal{A}y \in \mathbb{R}^n$ with the i -th entry

$$(\mathcal{A}yy)_i = \sum_{j,k \in N} a_{ijk} y_j y_k, \text{ and } (x\mathcal{A}y)_i = \sum_{j,k \in N} a_{jki} x_j y_k,$$

respectively, then λ is called a C -eigenvalue of \mathcal{A} , x and y are called associated left and right C -eigenvectors, respectively.

For C -eigenvalues and associated left and right C -eigenvectors of a piezoelectric-type tensor, Chen et al. in [1] also provided several related results, such as:

Property 1. For a piezoelectric-type tensor \mathcal{A} , there always exist C -eigenvalues of \mathcal{A} and associated left and right C -eigenvectors.

Property 2. Suppose that λ , x and y are a C -eigenvalue and its associated left and right C -eigenvectors of a piezoelectric-type tensor \mathcal{A} . Then

$$\lambda = x\mathcal{A}yy,$$

where $x\mathcal{A}yy = \sum_{i,j,k \in N} a_{ijk} x_i y_j y_k$. Furthermore, $(\lambda, x, -y)$, $(-\lambda, -x, y)$ and $(-\lambda, -x, -y)$ are also C -eigenvalues and their associated C -eigenvectors of \mathcal{A} .

Property 3. Suppose that λ^* is the largest C -eigenvalue of a piezoelectric-type tensor \mathcal{A} . Then

$$\lambda^* = \max \{x\mathcal{A}yy : x^T x = 1, y^T y = 1\}.$$

Property 2 and Property 3 provide theoretically the form to determine C -eigenvalues or the largest C -eigenvalue λ^* of \mathcal{A} . However, it is difficult to compute them in practice because determining x and y is not easy. So, we in this paper give some intervals to locate all C -eigenvalues of a piezoelectric-type tensor, and then give some upper bounds for the the largest C -eigenvalue. This can provide more information before calculating them out.

2. Main results

In this section, we give two intervals to locate all C -eigenvalues of a piezoelectric-type tensor. And the comparison of these two intervals are also established.

Theorem 1. *Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ be a piezoelectric-type tensor, and λ be a C -eigenvalue of \mathcal{A} . Then*

$$\lambda \in [-\rho, \rho], \quad (2)$$

where

$$\rho := \max_{i,j \in N} \left(R_i^{(1)}(\mathcal{A}) R_j^{(3)}(\mathcal{A}) \right)^{\frac{1}{2}},$$

$$R_i^{(1)}(\mathcal{A}) := \sum_{l,k \in N} |a_{ilk}| \text{ and } R_j^{(3)}(\mathcal{A}) := \sum_{l,k \in N} |a_{lkj}|.$$

Proof. Suppose that $x = (x_1, x_2, \dots, x_n)^T$ and $y = (y_1, y_2, \dots, y_n)^T$ are left and right C -eigenvectors corresponding to λ with $x^T x = 1$ and $y^T y = 1$. Let

$$|x_p| = \max_{i \in N} |x_i|, \text{ and } |y_q| = \max_{i \in N} |y_i|.$$

Then $0 < |x_p| \leq 1$ and $0 < |y_q| \leq 1$ because $x^T x = 1$ and $y^T y = 1$.

By considering the p -th equation of $\mathcal{A}yy = \lambda x$ in (1), we have

$$\lambda x_p = \sum_{j,k \in N} a_{pjk} y_j y_k, \quad (3)$$

and

$$\begin{aligned} |\lambda| |x_p| &\leq \sum_{j,k \in N} |a_{pjk}| |y_j| |y_k| \\ &\leq \sum_{j,k \in N} |a_{pjk}| |y_q| |y_q| \\ &\leq \sum_{j,k \in N} |a_{pjk}| |y_q|. \text{ (by } |y_q| \leq 1) \end{aligned}$$

Hence

$$|\lambda| |x_p| \leq R_p^{(1)}(\mathcal{A}) |y_q|. \quad (4)$$

On the other hand, by considering the q -th equation of $x\mathcal{A}y = \lambda y$ in (1), we have

$$\lambda y_q = \sum_{i,j \in N} a_{ijq} x_i y_j, \quad (5)$$

and

$$\begin{aligned} |\lambda| |y_q| &\leq \sum_{i,j \in N} |a_{ijq}| |x_i| |y_j| \\ &\leq \sum_{i,j \in N} |a_{ijq}| |x_p| |y_q| \\ &\leq \sum_{i,j \in N} |a_{ijq}| |x_p|. \quad (\text{by } |y_q| \leq 1) \end{aligned}$$

Hence

$$|\lambda| |y_q| \leq R_q^{(3)}(\mathcal{A}) |x_p|. \quad (6)$$

Multiplying (4) with (6) yields

$$|\lambda|^2 |x_p| |y_q| \leq R_p^{(1)}(\mathcal{A}) R_q^{(3)}(\mathcal{A}) |x_p| |y_q|,$$

consequently,

$$|\lambda| \leq (R_p^{(1)}(\mathcal{A}) R_q^{(3)}(\mathcal{A}))^{\frac{1}{2}}. \quad (7)$$

Note the facts that λ is a C -eigenvalue of \mathcal{A} if and only if $-\lambda$ is a C -eigenvalue of \mathcal{A} , and that a C -eigenvalue is real. Then

$$\lambda \in \left[- (R_p^{(1)}(\mathcal{A}) R_q^{(3)}(\mathcal{A}))^{\frac{1}{2}}, (R_p^{(1)}(\mathcal{A}) R_q^{(3)}(\mathcal{A}))^{\frac{1}{2}} \right] \subseteq [-\rho, \rho].$$

The conclusion follows. \square

From Theorem 1, we can obtain easily the following upper bound for the largest C -eigenvalue of a piezoelectric-type tensor.

Corollary 1. Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ be a piezoelectric-type tensor, and λ^* be the largest C -eigenvalue of \mathcal{A} . Then

$$\lambda^* \leq \rho.$$

Next we give another interval to locate all C -eigenvalues of a piezoelectric-type tensor. Before that some notation are given. For a subset S of N , denote

$$\Delta_S := \{(i, j) : i \in S \text{ or } j \in S\}$$

and

$$\overline{\Delta}_S := \{(i, j) : i \notin S \text{ and } j \notin S\}.$$

Given a piezoelectric-type tensor $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$, let

$$R_j^{\Delta_S, (3)}(\mathcal{A}) = \sum_{(l, k) \in \Delta_S} |a_{lkj}|, R_j^{\overline{\Delta}_S, (3)}(\mathcal{A}) = \sum_{(l, k) \in \overline{\Delta}_S} |a_{lkj}|,$$

where $R_j^{\Delta_S, (3)}(\mathcal{A}) = 0$ if $S = \emptyset$, and $R_j^{\overline{\Delta}_S, (3)}(\mathcal{A}) = 0$ if $S = N$. Obviously, $R_j^{(3)}(\mathcal{A}) = R_j^{\Delta_S, (3)}(\mathcal{A}) + R_j^{\overline{\Delta}_S, (3)}(\mathcal{A})$ for each $j \in N$.

Theorem 2. *Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ be a piezoelectric-type tensor, and λ be a C -eigenvalue of \mathcal{A} . And let S be a subset of N . Then*

$$\lambda \in [-\rho_S, \rho_S], \quad (8)$$

where

$$\rho_S := \max_{i, j \in N} \frac{1}{2} \left(R_j^{\Delta_S, (3)}(\mathcal{A}) + \left((R_j^{\Delta_S, (3)}(\mathcal{A}))^2 + 4R_i^{(1)}(\mathcal{A})R_j^{\overline{\Delta}_S, (3)}(\mathcal{A}) \right)^{\frac{1}{2}} \right).$$

Furthermore,

$$\lambda \in [-\rho_{\min}, \rho_{\min}], \quad (9)$$

where $\rho_{\min} := \min_{S \subseteq N} \rho_S$.

Proof. Similarly to the proof of Theorem 1, (4) and (5) hold. Furthermore, by (5) we have

$$\begin{aligned} |\lambda| |y_q| &\leq \sum_{i, j \in N} |a_{ijq}| |x_p| |y_q| \\ &= R_q^{(3)}(\mathcal{A}) |x_p| |y_q| \\ &= \left(R_q^{\Delta_S, (3)}(\mathcal{A}) + R_q^{\overline{\Delta}_S, (3)}(\mathcal{A}) \right) |x_p| |y_q| \\ &\leq R_q^{\Delta_S, (3)}(\mathcal{A}) |y_q| + R_q^{\overline{\Delta}_S, (3)}(\mathcal{A}) |x_p| \end{aligned}$$

Hence

$$(|\lambda| - R_q^{\Delta_{S,(3)}}(\mathcal{A})) |y_q| \leq R_q^{\overline{\Delta}_{S,(3)}}(\mathcal{A}) |x_p|. \quad (10)$$

Multiplying (4) with (10) yields

$$|\lambda| (|\lambda| - R_q^{\Delta_{S,(3)}}(\mathcal{A})) |x_p| |y_q| \leq R_p^{(1)}(\mathcal{A}) R_q^{\overline{\Delta}_{S,(3)}}(\mathcal{A}) |x_p| |y_q|,$$

consequently,

$$|\lambda| (|\lambda| - R_q^{\Delta_{S,(3)}}(\mathcal{A})) \leq R_p^{(1)}(\mathcal{A}) R_q^{\overline{\Delta}_{S,(3)}}(\mathcal{A}). \quad (11)$$

Solving (11) for $|\lambda|$ gives

$$|\lambda| \leq \frac{1}{2} \left(R_q^{\Delta_{S,(3)}}(\mathcal{A}) + \left((R_q^{\Delta_{S,(3)}}(\mathcal{A}))^2 + 4R_p^{(1)}(\mathcal{A}) R_q^{\overline{\Delta}_{S,(3)}}(\mathcal{A}) \right)^{\frac{1}{2}} \right).$$

By an analogous way of Theorem 1, we have

$$\lambda \in [-\rho_S, \rho_S]. \quad (12)$$

Furthermore, since (12) holds for any $S \subseteq N$, it follows that

$$\lambda \in \bigcap_{S \subseteq N} [-\rho_S, \rho_S] = \left[-\min_{S \subseteq N} \rho_S, \min_{S \subseteq N} \rho_S \right] = [-\rho_{min}, \rho_{min}].$$

The conclusion follows. \square

Note here that if $S = \emptyset$, then $R_j^{\Delta_{S,(3)}}(\mathcal{A}) = 0$ and $R_j^{\overline{\Delta}_{S,(3)}}(\mathcal{A}) = R_j^{(3)}(\mathcal{A})$ for any $j \in N$, which implies

$$\frac{1}{2} \left(R_j^{\Delta_{S,(3)}}(\mathcal{A}) + \left((R_j^{\Delta_{S,(3)}}(\mathcal{A}))^2 + 4R_i^{(1)}(\mathcal{A}) R_j^{\overline{\Delta}_{S,(3)}}(\mathcal{A}) \right)^{\frac{1}{2}} \right) = \left(R_i^{(1)}(\mathcal{A}) R_j^{(3)}(\mathcal{A}) \right)^{\frac{1}{2}}$$

consequently,

$$\rho_S = \rho.$$

Hence,

$$\rho_{min} = \min_{S \subseteq N} \rho_S \leq \rho.$$

This gives the comparison of the intervals in Theorem 1 and Theorem 2 as follows.

Theorem 3. Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ be a piezoelectric-type tensor, and λ be a C -eigenvalue of \mathcal{A} . Then

$$\lambda \in [-\rho_{min}, \rho_{min}] \subseteq [-\rho, \rho],$$

where ρ is defined in Theorem 1, and ρ_{min} is defined in Theorem 2.

Remark 1. Theorem 3 shows that the interval $[-\rho_{min}, \rho_{min}]$ captures all C -eigenvalues of a piezoelectric-type tensor precisely than the interval $[-\rho, \rho]$, although ρ_{min} needs more computations than ρ .

Similarly to Corollary 1, we can obtain easily the following upper bound for the largest C -eigenvalue of a piezoelectric-type tensor by Theorem 2.

Corollary 2. Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ be a piezoelectric-type tensor, and λ^* be the largest C -eigenvalue of \mathcal{A} . Then

$$\lambda^* \leq \rho_{min}.$$

3. Numerical examples

In this section, we give some examples to show the results obtained above. Consider the eight piezoelectric tensors in [1];

(I) The piezoelectric tensor \mathcal{A}_{VFeSb} [1, 4], with its entries

$$a_{123} = a_{213} = a_{312} = -3.68180677,$$

and other elements are zeros;

(II) The piezoelectric tensor \mathcal{A}_{SiO_2} [1, 2, 3], with its entries

$$a_{111} = -a_{122} = -a_{212} = -0.13685, \text{ and } a_{123} = -a_{213} = -0.009715,$$

and other elements are zeros;

(III) The piezoelectric tensor $\mathcal{A}_{Cr_2AgBiO_8}$ [1, 4], with its entries

$$a_{123} = a_{213} = -0.22163, \text{ } a_{113} = -a_{223} = 2.608665,$$

$$a_{311} = -a_{322} = 0.152485, \text{ and } a_{312} = -0.37153,$$

and other elements are zeros;

(IV) The piezoelectric tensor \mathcal{A}_{RbTaO_3} [1, 4], with its entries

$$a_{113} = a_{223} = -8.40955, \quad a_{222} = -a_{212} = -a_{211} = -5.412525, \\ a_{311} = a_{322} = -4.3031, \quad \text{and } a_{333} = -5.14766,$$

and other elements are zeros;

(V) The piezoelectric tensor \mathcal{A}_{NaBiS_2} [1, 4], with its entries

$$a_{113} = -8.90808, \quad a_{223} = -0.00842, \quad a_{311} = -7.11526, \\ a_{322} = -0.6222, \quad \text{and } a_{333} = -7.93831,$$

and other elements are zeros;

(VI) The piezoelectric tensor $\mathcal{A}_{LiBiB_2O_5}$ [1, 4], with its entries

$$a_{123} = 2.35682, \quad a_{112} = 0.34929, \quad a_{211} = 0.16101, \quad a_{222} = 0.12562, \\ a_{233} = 0.1361, \quad a_{213} = -0.05587, \quad a_{323} = 6.91074, \quad \text{and } a_{312} = 2.57812,$$

and other elements are zeros;

(VII) The piezoelectric tensor $\mathcal{A}_{KBi_2F_7}$ [1, 4], with its entries

$$a_{111} = 12.64393, \quad a_{122} = 1.08802, \quad a_{133} = 4.14350, \quad a_{123} = 1.59052, \\ a_{113} = 1.96801, \quad a_{112} = 0.22465, \quad a_{211} = 2.59187, \quad a_{222} = 0.08263, \\ a_{233} = 0.81041, \quad a_{223} = 0.51165, \quad a_{213} = 0.71432, \quad a_{212} = 0.10570, \\ a_{311} = 1.51254, \quad a_{322} = 0.68235, \quad a_{333} = -0.23019, \quad a_{323} = 0.19013, \\ a_{313} = 0.39030, \quad \text{and } a_{312} = 0.08381.$$

(VIII) The piezoelectric tensor \mathcal{A}_{BaNiO_3} [1, 4], with its entries

$$a_{113} = a_{223} = 0.038385, \quad a_{311} = a_{322} = 6.89822, \quad \text{and } a_{333} = 27.4628,$$

and other elements are zeros.

We now use the intervals in Theorem 1 and Theorem 2 to locate all C -eigenvalues of the eight tensors above, see Table 1. It is easy to see that for any C -eigenvalue λ ,

$$\lambda \in [-\rho_{min}, \rho_{min}] \subseteq [-\rho, \rho].$$

	\mathcal{A}_{VFeSb}	\mathcal{A}_{SiO_2}	$\mathcal{A}_{Cr_2AgBiO_8}$	\mathcal{A}_{RbTaO_3}	\mathcal{A}_{NaBiS_2}	$\mathcal{A}_{LiBiB_2O_5}$	$\mathcal{A}_{KBi_2F_7}$	\mathcal{A}_{BaNiO_3}
ρ	7.3636	0.2882	5.6606	30.0911	17.3288	15.2911	22.6896	38.8162
ρ_{min}	7.3636	0.2834	5.6606	23.5377	16.8548	12.3206	20.2351	35.3787
λ^*	4.2514	0.1375	2.6258	12.4234	11.6674	7.7376	13.5021	27.4628

Table 1. The intervals $[-\rho, \rho]$ and $[-\rho_{min}, \rho_{min}]$, and λ^* is the largest C -eigenvalue.

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